

ANALYZING THE WEYL-HEISENBERG FRAME IDENTITY

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ABSTRACT. In 1990, Daubechies proved a fundamental identity for Weyl-Heisenberg systems which is now called the Weyl-Heisenberg Frame Identity. WH-Frame Identity: If $g \in W(L^\infty, L^1)$, then for all continuous, compactly supported functions f we have:

$$\sum_{m,n} |\langle f, E_{mb} T_{na} g \rangle|^2 = \frac{1}{b} \sum_k \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) \sum_n g(t - na) \overline{g(t - na - k/b)} dt.$$

It has been folklore that the identity will not hold universally. We make a detailed study of the WH-Frame Identity and show: (1) The identity does not require any assumptions on ab (such as the requirement that $ab \leq 1$ to have a frame); (2) As stated above, the identity holds for all $f \in L^2(\mathbb{R})$; (3) The identity holds for all bounded, compactly supported functions if and only if $g \in L^2(\mathbb{R})$; (4) The identity holds for all compactly supported functions if and only if $\sum_n |g(x - na)|^2 \leq B$ a.e.; Moreover, in (2)-(4) above, the series on the right converges unconditionally; (5) In general, there are WH-frames and functions $f \in L^2(\mathbb{R})$ so that the series on the right does not converge (even symmetrically). We give necessary and sufficient conditions for it to converge symmetrically; (6) There are WH-frames for which the series on the right always converges symmetrically to give the WH-Frame Identity, but there are functions for which the series does not converge and we classify when the series converges for all functions $f \in L^2(\mathbb{R})$; (7) There are WH-frames for which the series always converges, but it does not converge unconditionally for some functions, and we classify when we have unconditional convergence for all functions f ; and (8) We show that the series converges unconditionally for all $f \in L^2(\mathbb{R})$ if g satisfies the CC-condition.

1. INTRODUCTION

In 1990, Daubechies [1] proved a fundamental identity for Weyl-Heisenberg systems, which is now called the **Weyl-Heisenberg Frame Identity** (or **WH-frame Identity** for short). This identity has been extensively used in the theory and has gone through some small improvements over time. It has been part of the folklore that the identity does not hold universally. But, until now, it has been a little mysterious as to exactly when and where can one be sure the identity holds. In this paper we give a detailed analysis of the

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WH-frame Identity and answer all the relevant questions completely.

Casazza, Christensen, and Janssen [3] made a detailed study of Weyl-Heisenberg frames, translation invariant systems and the Walnut representation of the frame operator. We will rely heavily here on these results and the relevant constructions from [3] using the Zak transform.

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2. PRELIMINARIES

In this section we will give the basic results needed throughout the paper. We use $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ to denote the natural numbers, integers, real numbers and complex numbers, respectively. A **scalar** is an element of \mathbb{R} or \mathbb{C} . Integration is always with respect to Lebesgue measure. $L^2(\mathbb{R})$ will denote the complex Hilbert space of square integrable functions mapping \mathbb{R} into \mathbb{C} . A bounded unconditional basis for a Hilbert space H is called a **Riesz basis**. That is, (f_n) is a Riesz basis for H if and only if there is an orthonormal basis (e_n) for H and an operator $T : H \rightarrow H$ defined by $T(e_n) = f_n$, for all n . We call (f_n) a **Riesz basic sequence** if it is a Riesz basis for its closed linear span.

In 1952, Duffin and Schaeffer [5] were working on some deep problems in non-harmonic Fourier series. This led them to define

Definition 2.1. *A sequence $(f_n)_{n \in \mathbb{Z}}$ of elements of a Hilbert space H is called a **frame** if there are constants $A, B > 0$ such that*

$$(2.1) \quad A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The numbers A, B are called the **lower** and **upper frame bounds** respectively. The frame is a **tight frame** if $A = B$ and a **normalized tight frame** if $A = B = 1$. A frame is **exact** if it ceases to be a frame when any one of its elements is removed. It is known that a frame is exact if and only if it is a Riesz basis. A non-exact frame is called **over-complete** in the sense that at least one vector can be removed from the frame and the remaining set of vectors will still form a frame for H (but perhaps with different frame bounds). If $f_n \in H$, for all $n \in \mathbb{Z}$, we call $(f_n)_{n \in \mathbb{Z}}$ a **frame sequence** if it is a frame for its closed linear span in H .

We will consider frames from the operator theoretic point of view. To formulate this approach, let (e_n) be an orthonormal basis for an infinite dimensional Hilbert space H and let $f_n \in H$, for all $n \in \mathbb{Z}$. We call the operator $T : H \rightarrow H$

given by $Te_n = f_n$ the **preframe operator** associated with (f_n) . Now, for each $f \in H$ and $n \in \mathbb{Z}$ we have $\langle T^*f, e_n \rangle = \langle f, Te_n \rangle = \langle f, f_n \rangle$. Thus

$$(2.2) \quad T^*f = \sum_n \langle f, f_n \rangle e_n, \quad \text{for all } f \in H.$$

By (2.2)

$$\|T^*f\|^2 = \sum_n |\langle f, f_n \rangle|^2, \quad \text{for all } f \in H.$$

It follows that the preframe operator is bounded if and only if (f_n) has a finite upper frame bound B . Comparing this to Definition 2.1 we have

Theorem 2.2. *Let H be a Hilbert space with an orthonormal basis (e_n) . Also let (f_n) be a sequence of elements of H and let $Te_n = f_n$ be the preframe operator. The following are equivalent:*

- (1) *(f_n) is a frame for H .*
- (2) *The operator T is bounded, linear and onto.*
- (3) *The operator T^* is an (possibly into) isomorphism called the **frame transform**.*

Moreover, (f_n) is a normalized tight frame if and only if the preframe operator is a quotient map (i.e. a co-isometry).

The dimension of the kernel of T is called the **excess** of the frame. It follows that $S = TT^*$ is an invertible operator on H , called the **frame operator**. Moreover, we have

$$Sf = TT^*f = T\left(\sum_n \langle f, f_n \rangle e_n\right) = \sum_n \langle f, f_n \rangle Te_n = \sum_n \langle f, f_n \rangle f_n.$$

A direct calculation now yields

$$\langle Sf, f \rangle = \sum_n |\langle f, f_n \rangle|^2.$$

Therefore, the **frame operator is a positive, self-adjoint invertible operator** on H . Also, the frame inequalities (2.1) yield that (f_n) is a frame with frame bounds $A, B > 0$ if and only if $A \cdot I \leq S \leq B \cdot I$. Hence, (f_n) is a normalized tight frame if and only if $S = I$. Also, a direct calculation yields

$$(2.3) \quad \begin{aligned} f = SS^{-1}f &= \sum_n \langle S^{-1}f, f_n \rangle f_n \\ &= \sum_n \langle f, S^{-1}f_n \rangle f_n \\ &= \sum_n \langle f, S^{-1/2}f_n \rangle S^{-1/2}f_n. \end{aligned}$$

We call $(\langle S^{-1}f, f_n \rangle)$ the **frame coefficients** for f . One interpretation of equation (2.3) is that $(S^{-1/2}f_n)$ is a normalized tight frame.

We will work here with a particular class of frames called Weyl-Heisenberg frames. To formulate these frames, we first need some notation. For a function f on \mathbb{R} we define the operators:

$$\begin{aligned} \text{Translation: } T_a f(x) &= f(x - a), \quad a \in \mathbb{R} \\ \text{Modulation: } E_a f(x) &= e^{2\pi i a x} f(x), \quad a \in \mathbb{R} \end{aligned}$$

We also use the symbol E_a to denote the **exponential function** $E_a(x) = e^{2\pi i a x}$. Each of the operators T_a, E_a are unitary operators on $L^2(\mathbb{R})$.

In 1946 Gabor [6] formulated a fundamental approach to signal decomposition in terms of elementary signals. This method resulted in **Gabor frames** or as they are often called today **Weyl-Heisenberg frames**.

Definition 2.3. *If $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we call $(E_{mb}T_{na}g)_{m,n \in \mathbb{Z}}$ a **Weyl-Heisenberg system** (**WH-system** for short) and denote it by (g, a, b) . We call g the **window function**.*

If the WH-system (g, a, b) forms a frame for $L^2(\mathbb{R})$, we call this a **Weyl-Heisenberg frame** (**WH-frame** for short). The numbers a, b are the **frame parameters** with a being the **shift parameter** and b being the **modulation parameter**. We will be interested in when there are finite upper frame bounds for a WH-system. We call this class of functions the **preframe functions** and denote this class by **PF**. It is easily checked that

Proposition 2.4. *The following are equivalent:*

- (1) $g \in \mathbf{PF}$.
- (2) *The operator*

$$Sf = \sum_n \langle f, E_{mb}T_{na}g \rangle E_{mb}T_{na}g,$$

is a well defined bounded linear operator on $L^2(\mathbb{R})$.

A family (g, a, b) with $g \in \mathbf{PF}$ is called a **preframe WH-system**. It is a simple calculation to check the following (see [4]):

Proposition 2.5. *Let $f, g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$.*

- (1) *We have*

$$\sum_{k \in \mathbb{Z}} f(t - k/b)g(t - k/b - na) \in L^1[0, 1/b].$$

- (2) *If $\sum_k |f(t - ka)|^2 \leq B$ then for all $n \in \mathbb{Z}$,*

$$\sum_{k \in \mathbb{Z}} f(t - ka)g(t - ka - n/b) \in L^2[0, 1/b].$$

- (3) *If $\sum_n |g(t - na)|^2 \leq B$ then $\sum_n |g(t - na)g(t - na - k/b)| \leq B$, for all $k \in \mathbb{Z}$.*

We next recall the **Wiener amalgam space** $W(L^\infty, L^1)$ which consists of all functions g so that for some $a > 0$ we have,

$$\|g\|_{W,a} = \sum_{n \in \mathbb{Z}} \|g \cdot \chi_{[an, a(n+1))}\|_\infty = \sum_{n \in \mathbb{Z}} \|T_{na} \cdot \chi_{[0,a)}\|_\infty < \infty.$$

It is easily checked that $W(L^\infty, L^1)$ is a Banach space with the above norm. Also, if $\|g\|_{W,a} < \infty$, for one $a > 0$, then this norm is finite for all $a > 0$.

To simplify some of the results we introduce some notation. For any $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we let for all $k \in \mathbb{Z}$,

$$G_k(t) = \sum_{n \in \mathbb{Z}} g(t - na) \overline{g(t - na - k/b)}.$$

In particular,

$$G_0(t) = \sum_{n \in \mathbb{Z}} |g(t - na)|^2.$$

Our main tool will be the proof of the WH-frame Identity due to Walnut [7]. He eliminated the need for the Poisson summation formula used by Daubechies in the original proof and obtained a more general result.

Theorem 2.6. (WH-Frame Identity.) *If $g \in W(L^\infty, L^1)$ and $f \in L^2(\mathbb{R})$ is continuous and compactly supported, then*

$$\begin{aligned} \sum_{n,m \in \mathbb{Z}} | \langle f, E_{mb} T_{na} g \rangle |^2 = \\ \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt = F_1(f) + F_2(f), \end{aligned}$$

where

$$F_1(f) = b^{-1} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt,$$

and

$$\begin{aligned} F_2(f) &= b^{-1} \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \\ &= b^{-1} \sum_{k \geq 1} 2 \operatorname{Re} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt. \end{aligned}$$

Proof. We are assuming that f is bounded and compactly supported so that all the summations, integrals and interchanges of these below are justified. We define

$$H_n(t) = \sum_k f(t - k/b) \overline{g(t - na - k/b)}.$$

Now, H_n is $1/b$ -periodic, $H_n \in L^2[0, 1/b]$ and

$$\int_R f \cdot \overline{E_{mb}T_{na}g(t)}dt = \int_R f(t)\overline{g(t-na)}e^{-2\pi imbt}dt = \int_0^{1/b} H_n(t)e^{-2\pi imbt}dt.$$

Since $(b^{1/2}E_{mb})_{m \in \mathbb{Z}}$ is an orthonormal basis for $L^2[0, 1/b]$, the Plancherel formula yields

$$\sum_m \left| \int_0^{1/b} H_n(t)e^{-2\pi imbt}dt \right|^2 = b^{-1} \int_0^{1/b} |H_n(t)|^2 dt.$$

Now we compute

$$\begin{aligned} \sum_n \sum_m | \langle f, E_{mb}T_{na}g \rangle |^2 &= \sum_n \sum_m \left| \int_R f(t)\overline{g(t-na)}e^{-2\pi imbt}dt \right|^2 \\ &= b^{-1} \sum_n \int_0^{1/b} \left| \sum_k f(t-k/b)\overline{g(t-na-k/b)} \right|^2 dt \\ &= b^{-1} \sum_n \int_0^{1/b} \sum_\ell \overline{f(t-\ell/b)}g(t-na-\ell/b) \cdot \sum_k f(t-k/b)\overline{g(t-na-k/b)}dt \\ &= b^{-1} \sum_n \sum_\ell \int_0^{1/b} \overline{f(t-\ell/b)}g(t-na-\ell/b) \cdot \sum_k f(t-k/b)\overline{g(t-na-k/b)}dt \\ &= b^{-1} \sum_n \int_R \overline{f(t)}g(t-na) \cdot \sum_k f(t-k/b)\overline{g(t-na-k/b)}dt \\ &= b^{-1} \sum_k \int_R \overline{f(t)}f(t-k/b) \cdot \sum_n g(t-na)\overline{g(t-na-k/b)}dt \\ &= b^{-1} \int_R |f(t)|^2 \cdot \sum_n |g(t-na)|^2 dt + \\ &\quad b^{-1} \sum_{k \neq 0} \int_R \overline{f(t)}f(t-k/b) \cdot \sum_n g(t-na)\overline{g(t-na-k/b)}dt. \end{aligned}$$

This completes the first part of the WH-Frame Identity. The equality in the last line follows by a simple change of variables. \square

To avoid “technicalities” we will say that the **WH-frame Identity holds for a function \mathbf{f}** to mean that the series on the left hand side sums to be finite and is equal to the right hand side sum which converges unconditionally. Later, we will discuss different forms of convergence for the right hand side of the WH-frame Identity.

As a consequence of the WH-frame Identity, Casazza, Christensen and Janssen [3] showed:

Proposition 2.7. *Let $a, b \in \mathbb{R}$ with $ab \leq 1$ and $g \in L^2(\mathbb{R})$ and assume that*

$$\sum_{k \in \mathbb{Z}} |G_k(t)|^2 \leq B, \quad a.e.$$

Then for all bounded, compactly supported functions $f \in L^2(\mathbb{R})$ the series

$$Lf = b^{-1} \sum_k (T_{k/b} f) G_k,$$

converges unconditionally in norm in $L^2(\mathbb{R})$. Moreover,

$$\langle Lf, f \rangle = \sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2.$$

Finally, if $g \in \mathbf{PF}$, so that the series

$$Sf = \sum_{m, n} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g,$$

also converges unconditionally in $L^2(\mathbb{R})$, we have that $Lf = Sf$.

We will also make use of the CC-condition from [2].

Theorem 2.8 (CC-Condition). *If $g \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$ and*

$$(CC) \quad \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g(t - na) \overline{g(t - na - k/b)} \right| = \sum_{k \in \mathbb{Z}} |G_k(t)| \leq B, \quad a.e.,$$

then $g \in \mathbf{PF}$. Moreover, if we also have

$$(2.4) \quad \sum_{k \neq 0} |G_k(t)| \leq (1 - \epsilon) G_0(t) \quad a.e.,$$

for some $0 < \epsilon < 1$, then (g, a, b) is a WH-frame.

We will need a special representation of the frame operator for WH-frames due to Walnut [7].

Theorem 2.9. *Let $a, b > 0$ and $g \in W(L^\infty, L^1)$ be given. For each $f \in L^2(\mathbb{R})$, the sum Sf converges and is given by*

$$Sf = \frac{1}{b} \sum_{k \in \mathbb{Z}} T_{k/b} f \cdot G_k.$$

The series in Theorem 2.9 is called the **Walnut representation of the frame operator**. The precise conditions under which the Walnut representation converges to the frame operator are quite delicate and were studied in detail in [3].

3. BOUNDED, COMPACTLY SUPPORTED FUNCTIONS AND THE WH-FRAME IDENTITY

We start with a simple observation.

Proposition 3.1. *If $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ is bounded and compactly supported, then $g \in PF$.*

Proof. First, assume that g is supported on $[0, a]$. Since g is bounded above and compactly supported, there is a constant B so that

$$(3.1) \quad \sum_{k \in \mathbb{Z}} |g(t - k/b)|^2 \leq B.$$

We define the preframe operator $L : \ell_2 \otimes \ell_2 \rightarrow L^2(\mathbb{R})$ by

$$L\left(\sum_{m,n \in \mathbb{Z}} a_{mn} e_{mn}\right) = \sum_{m,n \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g,$$

where (e_{mn}) is the natural orthonormal basis of $\ell_2 \otimes \ell_2$. We need to show that L is a bounded operator. By our assumption on the support of g , we see that $(T_{na} g)_{n \in \mathbb{Z}}$ are disjointly supported functions. Hence,

$$(3.2) \quad \|L\left(\sum_{m,n \in \mathbb{Z}} a_{mn} e_{mn}\right)\|^2 = \sum_{n \in \mathbb{Z}} \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2.$$

Applying inequality 3.1 above at the appropriate step, we have

$$\begin{aligned} \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2 &= \int_{\mathbb{R}} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g(t) \right|^2 dt \\ &= \int_0^{1/b} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} \right|^2 \sum_{k \in \mathbb{Z}} |g(t - k/b - na)|^2 dt \\ &\leq B \int_0^{1/b} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} \right|^2 dt \\ &= B \sum_{m \in \mathbb{Z}} |a_{mn}|^2. \end{aligned}$$

It follows from equation (3.2),

$$\left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{mb} T_{na} g \right\|^2 \leq \sum_{mn \in \mathbb{Z}} |a_{mn}|^2.$$

Hence, L is a bounded operator.

For the general case, we observe that g can be written as a finite sum, say k , of translates of functions supported on $[0, a]$ and so the preframe function is also bounded in this case by $k\|L\|$. \square

Corollary 3.2. *If $g \in L^2(\mathbb{R})$, then for every bounded, compactly supported function f on \mathbb{R} , we have*

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 < \infty.$$

Proof. By Proposition 3.1, if f is bounded and compactly supported then $(E_{mb}T_{na}f)_{m,n \in \mathbb{Z}}$ has a finite upper frame bound, say B . Now

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 &= \sum_{m,n \in \mathbb{Z}} |\langle T_{-na}E_{-mb}f, g \rangle|^2 = \\ \sum_{m,n \in \mathbb{Z}} |e^{-2\pi imb(x-na)} \langle E_{mb}T_{na}f, g \rangle|^2 &= \sum_{m,n \in \mathbb{Z}} |\langle E_{mb}T_{na}f, g \rangle|^2 \leq B. \end{aligned}$$

□

We now present the main result of this section.

Theorem 3.3. *Let g be a measurable function on \mathbb{R} . The following are equivalent:*

- (1) $g \in L^2(\mathbb{R})$.
- (2) *The WH-frame Identity holds for all bounded, compactly supported functions f on \mathbb{R} .*

Proof. (1) \Rightarrow (2): We assume that f is supported on $[-N, N]$ and bounded above by B . For a fixed $n \in \mathbb{Z}$ we consider the $1/b$ -periodic function

$$H_n(t) = \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)}.$$

Now, the above sum only has $2N$ non-zero terms for each $t \in \mathbb{R}$. So we can easily follow Walnut's proof the the WH-frame Identity line for line interchanging the (now finite) sums and integrals until we arrive at:

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 &= \\ \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} g(t - na) \cdot \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)} &= \\ \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_{-N}^N \overline{f(t)} g(t - na) \cdot \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)} & \end{aligned}$$

To finish the identity, we just need to justify interchanging the infinite sum over n with the finite sum over k . To justify this we observe that:

$$\sum_{n,k} |f(t)| |g(t - na)| |f(t - k/b)| |f(t - na - k/b)| \leq$$

$$B^2 \sum_{k=-N}^N \sum_{n \in \mathbb{Z}} |g(t - na)g(t - na - k/b)|.$$

By Proposition 2.5, we have that

$$\sum_{n \in \mathbb{Z}} |g(t - na)g(t - na - k/b)| \in L^1[0, a],$$

and hence

$$\sum_{n \in \mathbb{Z}} |g(t - na)g(t - na - k/b)| \in L^1[-N, N].$$

Therefore, we justify the needed interchange of sums and sums with integrals by the Lebesgue Dominated Convergence Theorem.

(2) \Rightarrow (1): We do this by contradiction. If g is not square integrable on \mathbb{R} , then

$$\|g\|^2 = \int_0^a \sum_{n \in \mathbb{Z}} |f(t - na)|^2 dt = \infty.$$

Hence, there is some interval I of length c with $0 < c < 1/b$ so that

$$\int_I \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \infty.$$

If we let $f = \chi_I$, then the right hand side of the WH-frame Identity becomes

$$\int_{\mathbb{R}} |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \int_I \sum_{n \in \mathbb{Z}} |g(t - na)|^2 dt = \infty.$$

So the right hand side of the WH-frame identity is not a finite unconditionally convergent series. i.e. The WH-frame Identity fails. \square

4. COMPACTLY SUPPORTED FUNCTIONS AND THE WH-FRAME IDENTITY

In this section we will drop the hypotheses that our function f has to be bounded and discover necessary and sufficient conditions for the WH-frame Identity to hold. The conditions are a little stronger than those required for bounded, compactly supported functions.

Theorem 4.1. *Let g be a measurable function on \mathbb{R} . The following are equivalent:*

(1) *There is a constant $B > 0$ so that*

$$\sum_{n \in \mathbb{Z}} |g(t - na)|^2 \leq B, \quad a.e.$$

(2) *The WH-frame Identity holds for all compactly supported functions f on \mathbb{R} .*

Proof. (1) \Rightarrow (2): If f is compactly supported, we see immediately that the sum over k in the right hand side of the WH-frame identity is a finite sum. So let $f_\ell(t) = f(t)$ if $|f(t)| \leq \ell$ and zero otherwise. Now, by Theorem 3.3 the WH-frame identity holds for all f_ℓ . That is, for all $\ell \in \mathbb{Z}$ we have

$$\sum_{m,n \in \mathbb{Z}} | \langle f_\ell, E_{mb} T_{na} g \rangle |^2 = \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f_\ell(t)} f_\ell(t - k/b) G_k(t) dt$$

Now we will finish the proof in three steps.

Step 1: We show that

$$\begin{aligned} & \sum_{m,n} | \langle f, E_{mb} T_{na} g \rangle |^2 = \\ & = b^{-1} \sum_n \int_0^{1/b} \left| \sum_k f(t - k/b) \overline{g(t - na - k/b)} \right|^2 dt. \end{aligned}$$

To prove Step 1, we let for a fixed $n \in \mathbb{Z}$,

$$H_n(t) = \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - na - k/b)}.$$

Since the sum on the right hand side above is finite, we can copy the first steps of the Walnut proof of the WH-frame Identity to the point of the identity given in Step 1.

Step 2: We show,

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \sum_n \int_0^{1/b} \left| \sum_k f_\ell(t - k/b) \overline{g(t - na - k/b)} \right|^2 dt = \\ & = b^{-1} \sum_n \int_0^{1/b} \left| \sum_k f(t - k/b) \overline{g(t - na - k/b)} \right|^2 dt = \\ & \quad \sum_{m,n} | \langle f, E_{mb} T_{na} g \rangle |^2. \end{aligned}$$

For step 2, choose an N so that for all $t \in [0, 1/b]$, $f(t - k/b) = 0$ for all $|k| > N$. Hence, for all $t \in [0, 1/b]$ we have

$$\begin{aligned} & \sum_n \left| \sum_k f_\ell(t - k/b) \overline{g(t - na - k/b)} \right|^2 \leq \\ & \sum_{k=-N}^N |f_\ell(t - k/b)|^2 \sum_n \sum_{k=-N}^N |g(t - na - k/b)|^2 \leq \end{aligned}$$

$$\sum_{k=-N}^N |f(t - k/b)|^2 2NB^2 \in L^1[0, 1/b].$$

So, Step 2 follows by the Lebesgue Dominated convergence Theorem. The following step will complete the proof.

Step 3:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f_{\ell}(t)} f_{\ell}(t - k/b) G_k(t) dt = \\ \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt \end{aligned}$$

For Step 3, note that $\text{support } f_{\ell} \subset \text{support } f_{\ell+1} \subset \text{support } f$. Hence, for k fixed we have:

$$|f_{\ell}(t)| |T_{k/b} f_{\ell}(t)| |G_k(t)| \uparrow |f(t)| |T_{k/b} f(t)| |G_k(t)|.$$

Also, by assumption $|G_k(t)| \leq B^2$. Since $f \in L^2(\mathbb{R})$ this implies

$$|f(t)| |T_{k/b} f(t)| |G_k(t)| \in L^1(\mathbb{R}).$$

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{1}{b} \int_{\mathbb{R}} \overline{f_{\ell}(t)} f_{\ell}(t - k/b) G_k(t) dt = \\ \frac{1}{b} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt. \end{aligned}$$

Finally, since the right hand side of the WH-frame identity has only a finite number of non-zero k 's (and the same ones for f and all f_{ℓ}), we have the equality in Step 3 and unconditional convergence in the right hand side of the Identity.

(2) \Rightarrow (1): For any f supported on an interval of length $1/b$, we are assuming the WH-frame Identity holds. But, $F_2(f) = 0$ for all such f . So

$$F_1(f) = b^{-1} \int_{\mathbb{R}} |f(t)|^2 \sum_n |g(t - na)|^2 dt < \infty.$$

This implies that G_0 is bounded. To see this, let $I = [c, d]$ be any interval of length $< 1/b$. It suffices to show that G_0 is bounded here since G_0 is a-periodic means it is bounded if it is bounded on all intervals of any fixed length. Let

$$A_n = \{t \in I : |G_0(t)| \leq n\}.$$

Let $T_n : L^2[c, d] \rightarrow L^2[c, d]$ be given by $T_n f = \chi_{A_n} f \cdot \sqrt{G_0}$. The T_n are bounded linear operators and the family is pointwise bounded by the above. Hence they are uniformly bounded and so

$$Tf = f \cdot G_0$$

is a bounded linear operator. But the norm of this “multiplication” operator is $\text{ess sup } |G_0(t)|$.

□

We remark that we could simplify the proof of Theorem 4.1 if $g \in \text{PF}$. For in this case we use the frame operator S to get some of the needed convergence. For example, in this case we would observe:

$$\sum_{m,n \in \mathbb{Z}} | \langle f, E_{mb} T_{na} g \rangle |^2 = \langle Sf, f \rangle,$$

while

$$\lim_{\ell \rightarrow \infty} \langle Sf_\ell, f_\ell \rangle = \langle Sf, f \rangle.$$

5. TYPES OF CONVERGENCE OF THE WH-FRAME IDENTITY

Here we will consider when the WH-frame Identity holds with a stipulation on the type of convergence of the infinite series on the right hand side of the identity. These results are variations on results of Casazza, Christensen and Janssen [3]. To extend the results of [3] we need a well known fact which is really the Polarization Identity for H .

Proposition 5.1. *For any operator T on a complex Hilbert space H we have for all $x, y \in H$:*

$$\begin{aligned} 4 \langle Tx, y \rangle &= \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + \\ &\quad i \langle T(x+iy), x+iy \rangle - i \langle T(x-iy), x-iy \rangle. \end{aligned}$$

Corollary 5.2. *If T is an operator on a complex Hilbert space H , then*

$$\|T\| \leq 2 \sup \{ | \langle Tf, f \rangle | : \|f\| \leq 1 \}.$$

We need some notation for checking the convergence of the series in the WH-frame Identity.

Definition 5.3. *Let $g \in L^2(R)$ satisfy $G_0(t) \leq B$ a.e. For any $f \in L^2(R)$, and any $K, L \in \mathbb{Z}$, we let*

$$S_{K,L}f(t) = \sum_{k=-L}^K f(t - k/b) G_k(t),$$

where as usual $G_k(t) = \sum_n g(t - na) \overline{g(t - na - k/b)}$. We also let $S_K = S_{K,K}$. If $M \subset \mathbb{Z}$ with $|M| < \infty$, define

$$S_M f(t) = \sum_{k \in M} f(t - k/b) G_k(t).$$

If $\lim_{K \rightarrow \infty} S_K f$ exists, we say that **the Walnut series for f converges symmetrically** - and this can be in either the norm or the weak topology - and we say **the Walnut series for f converges** when $\lim_{K, L \rightarrow \infty} S_{K, L} f$ exists. Now we give an extension of Theorem 5.2 from [3]. The notation can be found in Section 3.1.

Theorem 5.4 (Casazza/Christensen/Janssen). *Let $a, b \in \mathbb{R}$, $g \in L^2(\mathbb{R})$ and $G_0(t) \leq B$ a.e. The following are equivalent:*

- (1) *The Walnut series converges in norm symmetrically for every $f \in L^2(\mathbb{R})$.*
- (2) *The Walnut series converges weakly symmetrically for every $f \in L^2(\mathbb{R})$.*
- (3) *We have $\sup_K \|S_K\| < \infty$.*

Moreover, in this case the WH-system (g, a, b) has a finite upper frame bound and the Walnut series converges symmetrically to Sf , for all $f \in L^2(\mathbb{R})$.

We now extend this result slightly.

Theorem 5.5. *Let $a, b \in \mathbb{R}$, $g \in L^2(\mathbb{R})$ and $G_0(t) \leq B$ a.e. The following are equivalent:*

- (1) *The Walnut series converges in norm symmetrically for every $f \in L^2(\mathbb{R})$.*
- (2) *We have for all $f \in L^2(\mathbb{R})$,*

$$\lim_{K \rightarrow \infty} \langle S_K f, f \rangle = \langle Sf, f \rangle,$$

- (3) *We have for all $f \in L^2(\mathbb{R})$,*

$$\sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 = \lim_{K \rightarrow \infty} \frac{1}{b} \sum_{k=-K}^K \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt.$$

Proof. (1) \Leftrightarrow (2): If we assume (2), we easily obtain from Proposition 5.1 that $\lim_{K \rightarrow \infty} \langle S_K f, h \rangle = \langle Sf, h \rangle$, for all $f, h \in L^2(\mathbb{R})$. i.e. The Walnut series converges weakly symmetrically, and hence symmetrically in norm. The converse is obvious.

(3) \Rightarrow (2): The right hand side of (3) is:

$$\lim_{K \rightarrow \infty} \frac{1}{b} \sum_{k=-K}^K \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt = \frac{1}{b} \lim_{K \rightarrow \infty} \langle S_K f, f \rangle.$$

This implies that

$$\lim_{K \rightarrow \infty} \langle S_K f, f \rangle \text{ exists}$$

for all $f \in L^2(\mathbb{R})$. By Proposition 5.1, it follows that $(S_K f)$ is weakly symmetrically convergent (and hence symmetrically norm convergent by Theorem 5.4) for all $f \in L^2(\mathbb{R})$. Hence, the (S_K) are uniformly bounded operators. By

Proposition 2.7, the right hand side of (3) converges to $\langle Sf, f \rangle$ unconditionally on a dense subset of $L^2(\mathbb{R})$, and since the operators S_K are uniformly bounded, we have the equality in (2) for all $f \in L^2(\mathbb{R})$.

(1) \Rightarrow (3): By (1), the limit on the right hand side of (3) converges for all f and to $\langle Sf, f \rangle$. By Theorem 5.4, (g, a, b) has a finite upper frame bound. Now,

$$\begin{aligned} \lim_{K \rightarrow \infty} \frac{1}{b} \sum_{k=-K}^K \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt &= \lim_{K \rightarrow \infty} \langle S_K f, f \rangle = \\ &= \langle Sf, f \rangle = \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2. \end{aligned}$$

□

In [3] (Example 5.4) it is shown that there are WH-frames $(g, 1, 1)$ so that for some function $f \in L^2(\mathbb{R})$, the Walnut series for f does not converge symmetrically. Combined with Theorem 5.5 we obtain,

Corollary 5.6. *There is a WH-frame $(g, 1, 1)$ and a function $f \in L^2(\mathbb{R})$ so that the WH-frame identity fails for this f in the sense that the series on the right hand side of the WH-frame Identity does not converge symmetrically for this f .*

Next we generalize another result, Theorem 5.5, from [3].

Theorem 5.7. *Let $ab \in \mathbb{R}$ and $g \in PF$. The following are equivalent:*

- (1) *The Walnut series converges in norm for every $f \in L^2(\mathbb{R})$.*
- (2) *The Walnut series converges weakly for every $f \in L^2(\mathbb{R})$.*
- (3) *We have that $\sup_{K,L} \|S_{K,L}\| < \infty$.*
- (4) *We have for all $f \in L^2(\mathbb{R})$,*

$$\lim_{K,L \rightarrow \infty} \langle S_{K,L} f, f \rangle = \langle Sf, f \rangle.$$

- (5) *We have for all $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 &= \\ \lim_{K,L \rightarrow \infty} \frac{1}{b} \sum_{k=-L}^K \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) G_k(t) dt. \end{aligned}$$

Proof. The equivalence of (1) – (3) is due to Casazza, Christensen, and Janssen ([3], Theorem 5.5). The rest of the proof follows line by line the proof of our Theorem 5.5 above, just replacing symmetric convergence by convergence at each step. □

Again, in [3] (Example 5.7) it is shown that there is a WH-frame $(g, 1, 1)$ for which the Walnut series converges symmetrically for every $f \in L^2(\mathbb{R})$, but there is an $h \in L^2(\mathbb{R})$ for which the Walnut series does not converge in norm (or weakly). Combined with Theorem 5.7, we have,

Corollary 5.8. *There is a WH-frame $(g, 1, 1)$ for which the WH-frame Identity holds for all $f \in L^2(\mathbb{R})$ in the sense that the series on the right hand side of the identity converges symmetrically for all $f \in L^2(\mathbb{R})$ and we have equality in the identity. However, there is an $h \in L^2(\mathbb{R})$ for which the series on the right hand side of the WH-frame Identity does not converge.*

Our next theorem again generalizes a result (Theorem 6.1) from [3] and the proof follows along the lines of the proof of Theorem 5.5.

Theorem 5.9. *Let $a, b \in \mathbb{R}$ and $g \in PF$. The following are equivalent:*

- (1) *The Walnut series converges weakly unconditionally for every $f \in L^2(\mathbb{R})$.*
- (2) *The Walnut series converges unconditionally in norm for every $f \in L^2(\mathbb{R})$.*
- (3) *We have $\sup_{M \subset \mathbb{Z}, |M| < \infty} \|S_M\| < \infty$.*
- (4) *We have that the series*

$$\sum_k \langle (T_{k/b}f)G_k, f \rangle$$

converges unconditionally to $\langle Sf, f \rangle$, for all $f \in L^2(\mathbb{R})$.

- (5) *The WH-frame Identity holds and the series on the right hand side converges unconditionally for all $f \in L^2(\mathbb{R})$.*

In [3] (Example 6.3) it is shown that there is a WH-frame $(g, 1, 1)$ so that for every $f \in L^2(\mathbb{R})$ the Walnut series for f converges in norm, but there is some $h \in L^2(\mathbb{R})$ for which the Walnut series does not converge unconditionally. Combined with Theorem 5.9 we have,

Corollary 5.10. *There is a WH-frame $(g, 1, 1)$ so that for all $f \in L^2(\mathbb{R})$, the series on the right hand side of the WH-frame Identity converges and is equal to the left hand side. However, there is a function $h \in L^2(\mathbb{R})$ so that the series on the right hand side of the WH-frame Identity does not converge unconditionally.*

Casazza, Christensen and Janssen [3] Theorem 6.5 have shown that if (g, a, b) satisfies the CC-condition, then for all $f \in L^2(\mathbb{R})$, the Walnut series converges unconditionally. Also, it is immediate that if $g \in W(L^\infty, L^1)$, then (g, a, b) satisfies the CC-condition for all $a, b \in \mathbb{R}$. These results, combined with Theorem 5.9 yields,

Corollary 5.11. *If (g, a, b) satisfies the CC-condition, then the WH-frame Identity holds for all $f \in L^2(\mathbb{R})$ and the series converges unconditionally. In particular, if $g \in W(L^\infty, L^1)$ then the WH-frame Identity holds for all $f \in L^2(\mathbb{R})$ and the series converges unconditionally.*

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